PHYSICAL REVIEW C VOLUME 55, NUMBER 4 APRIL 1997

Neutron-proton correlations in an exactly solvable model

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(Received 29 October 1996)

We examine isovector and isoscalar neutron-proton correlations in an exactly solvable model based on the algebra SO(8). We look particularly closely at Gamow-Teller strength and double β decay, both to isolate the effects of the two kinds of pairing and to test two approximation schemes: the renormalized neutron-proton quasiparticle random phase approximation (QRPA) and generalized BCS theory. When isoscalar pairing correlations become strong enough a phase transition occurs and the dependence of the Gamow-Teller β^+ strength on isospin changes in a dramatic and unfamiliar way, actually increasing as neutrons are added to an $N=Z$ core. Renormalization eliminates the well-known instabilities that plague the QRPA as the phase transition is approached, but only by unnaturally suppressing the isoscalar correlations. Generalized BCS theory, on the other hand, reproduces the Gamow-Teller strength more accurately in the isoscalar phase than in the usual isovector phase, even though its predictions for energies are equally good everywhere. It also mixes $T=0$ and $T=1$ pairing, but only on the isoscalar side of the phase transition. [S0556-2813(97)02704-0]

PACS number(s): 21.60.Fw, 21.30.Fe, 23.40.Hc

I. INTRODUCTION

Pairing correlations are an important feature of nuclear structure [1]. In heavy nuclei such correlations between neutrons and protons are usually neglected on the grounds that the two Fermi levels are far apart. In nuclei with $N\approx Z$, however, the Fermi levels are close and neutron-proton (np) pairing correlations can be expected to play a significant role in nuclear structure and decay (for a review of work on np pairing theory see Ref. [2]). The importance of these proton-rich nuclei in astrophysical nucleosynthesis makes it vital that the np correlations are well understood, and upcoming experiments with radioactive beams will soon test our understanding.

Taking np correlations seriously complicates the usual treatment of pairing, which stresses the interaction of like particles in time-reversed orbits, i.e., the formation of pp and nn pairs. As has been known for some time [2,3], generalizing this picture raises at least two issues. First, the pp, nn, and np isovector pairs must all be treated on an equal footing so that isospin symmetry is respected as much as possible [3,4]. Second, the competition between two kinds of np pairing — isovector and isoscalar ($T=1$ and $T=0$) — must be taken into account. This issue apparently arises even in nuclei with $N>Z$, where np pairing is by most measures small. For example, the rate of two-neutrino double-β decay within the np quasiparticle random phase approximation (QRPA) is extremely sensitive to the strength of isoscalar particle-particle (i.e., pairing) correlations, making reliable calculations difficult. When these correlations become strong enough the method fails even to give finite answers.

The breakdown in the QRPA signals an impending phase transition. Is it real or an artifact of the assumption underlying the approximation that the ground state contains no np correlations? What are the properties of the ''isoscalar phase,'' if it is real? Is the renormalized QRPA (RQRPA) [5,6], in which solutions are more stable, a good way to handle the breakdown/phase transition? To what extent can generalized BCS theory, a scheme for treating np pairing on a more equal footing with nn and pp pairing and reviewed in Ref. [2], quantify the interplay between the two phases? We address these questions here in a solvable model that incorporates both isovector and isoscalar pairing, making it considerably richer than the more schematic models (e.g., the Lipkin model [7]) typically used for this kind of study.

The structure of this paper is as follows. In Sec. II we describe the model and its analytic solution for energies and Gamow-Teller β-decay matrix elements, stressing the existence of two limiting solutions corresponding to pure isovector or isoscalar pairing. We show that in the isoscalar phase, charge-changing processes have counterintuitive features. Section III contains an outline of the QRPA and RQRPA as realized in the solvable model, and applies them to single and double β decay to test the quality of the approximations. In Sec. IV we describe generalized BCS theory for SO(8), again with emphasisis on β-decay strengths (though we also examine ground-state energies), and again test the reliability of the approximation scheme. Section V is a conclusion.

II. THE MODEL AND ITS EXACT SOLUTION

We consider a set of degenerate single-particle orbitals, characterized by $l,s=1/2,t=1/2$. The total number of single-particle states is $\Omega=\Sigma_l(2l+1)$. We make the model solvable by building a basis entirely from $L=0$ operators: $S^\dagger_\nu$, which creates pairs with spin $S=0$ and isospin $T=1$ (with projection ν), and $P^\dagger_\mu$, which creates pairs with $S=1$ and

$T=0$ (spin projection μ). Together with the one body operators that generate SU(4) — the total spin $\vec{S}$, the isospin $\vec{T}$, and the operator $\mathcal{F}^{\mu}_{\nu}=\Sigma_i \sigma(i)_{\mu}\tau(i)_{\nu}$ — the $L=0$ pair creation and annihilation operators form the algebra SO(8) [8]. The physics associated with this model has been studied previously [8–10], but with emphasis on energy levels; here our focus will include charge-changing decay.

The most general Hamiltonian invariant under SO(8), omitting terms such as $\vec{S}\cdot\vec{S}$ and $\vec{T}\cdot\vec{T}$ that affect energies but not wave functions, depends on three parameters and has the form [9]

$$H=-\frac{g(1+x)}{2}\sum_{\nu} S^{\dagger}_{\nu}S_{\nu}-\frac{g(1-x)}{2}\sum_{\mu} P^{\dagger}_{\mu}P_{\mu}+g_{\rm ph}\mathcal{F}^{\mu\dagger}_{\nu}\mathcal{F}^{\mu}_{\nu}. \tag{1}$$

The first term in the Hamiltonian corresponds to isovector spin-0 pairing, the second represents isoscalar spin-1 pairing, and the last is a (primarily) particle-hole force in the $T=1$ $S=1$ channel.

In certain important limits, analytic expressions for energies and wave functions have been derived. If $x=1$ and $g_{\rm ph}=0$, the Hamiltonian [8] conserves an SO(5) subalgebra and corresponds to ''standard'' spin-singlet isovector pairing, with np pairs treated on an equal footing with like-particle pairs [11,12]. The eigenstates, characterized by the number of nucleon pairs $\mathcal{N}$ (we consider only nuclei with an even number of nucleons), the isospin T, and the singlet-pairing seniority v_s, have energies

$$E(v_s,T)=-\frac{g}{8}[(2\mathcal{N}-v_s)(4\Omega+6-2\mathcal{N}-v_s)-4T(T+1)]. \tag{2}$$

Similarly, for $x=-1$ and $g_{\rm ph}=0$ the exact solutions are characterized by the spin S and the triplet-pairing seniority v_t, and an analogous formula applies with $T\rightarrow S, v_s\rightarrow v_t$. This is the ''isoscalar phase'' that will cause the breakdown of the QRPA. Finally, if $x=0$ the Hamiltonian is invariant under SU(4). The eigenstates are then labeled by a quantum number λ corresponding to the irreducible SU(4) representation $[\lambda,\lambda,0]$ as well as by S and T, and the eigenvalues are

$$E(\lambda,S,T)=-\frac{g}{4}[2\mathcal{N}(\Omega+3)-\mathcal{N}^2-\lambda(\lambda+4)] \\ +g_{\rm ph}[\lambda(\lambda+4)-S(S+1)-T(T+1)]. \tag{3}$$

$S+T$ must be even if $\mathcal{N}$ is even and odd otherwise (this is true no matter what the Hamiltonian). The quantum number λ has values $\lambda=S+T, S+T+2, \ldots, \lambda_{\rm Max}$, where $\lambda_{\rm Max}=\mathcal{N}$ if $\mathcal{N}\leqslant\Omega$ and $2\Omega-\mathcal{N}$ otherwise.

The eigenvalues and eigenstates of the general Hamiltonian in Eq. (1) can be obtained by diagonalizing in the SU(4) basis. The matrices are tridiagonal and have very small dimension. Expressions for the matrix elements (with several typos) appear in Ref. [9], which, however, ignores the particle-hole interaction. The same model with the particle-hole interaction included was solved approximately in Ref. [13].

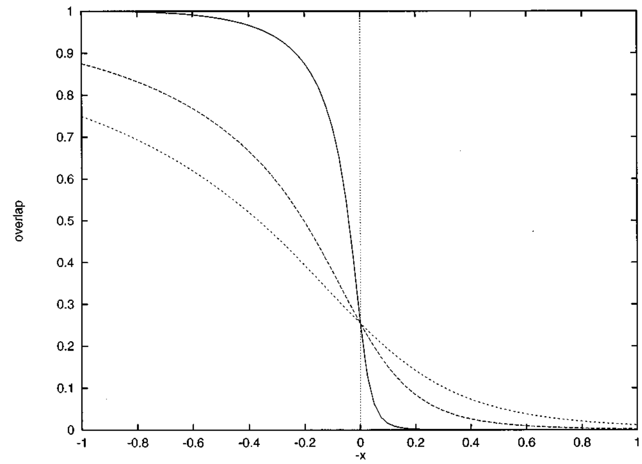

FIG. 1. Overlaps between the ground state and the pure isovector spin-singlet paired state vs the parameter x in the Hamiltonian of Eq. (1), for $\Omega=12, \mathcal{N}=10$ and $S=T=0$ (solid line), $S=0, T=2$ (long dashes), and $S=0, T=4$ (short dashes). Here and in Figs. 2, 4, 5, and 10 the quantity $-x$ is used on the abscissa axis so that the standard isovector phase is on the left.

By varying the parameter x one can study the phase transition from the standard spin-singlet isovector pairing phase, through the SU(4) Wigner supermultiplet phase, into the spin-triplet isoscalar pairing phase. In Fig. 1 we show the effects of this transition (with $g_{\rm ph}$ fixed at zero) on the overlap between the ground state and the ground state of the standard spin-singlet paired system with $x=1$ and $g_{\rm ph}=0$. (The abcissa is labeled by $-x$, so that the more familiar isovector phase is on the left.) The change produced by finite $g_{\rm ph}$ is illustrated in Fig. 2; when $g_{\rm ph}>0$ increases, the Hamiltonian more nearly conserves SU(4) symmetry and the phase transition becomes less pronounced. The overlap is an obvious ''order parameter'' in the model, and its point of inflection locates the phase transition. In even-even systems ($S=0$, T even) this point shifts to the right from the SU(4)-limit value $x=0$ as T increases. The reason is that any excess neutrons are necessarily in isovector pairs, making the tran-

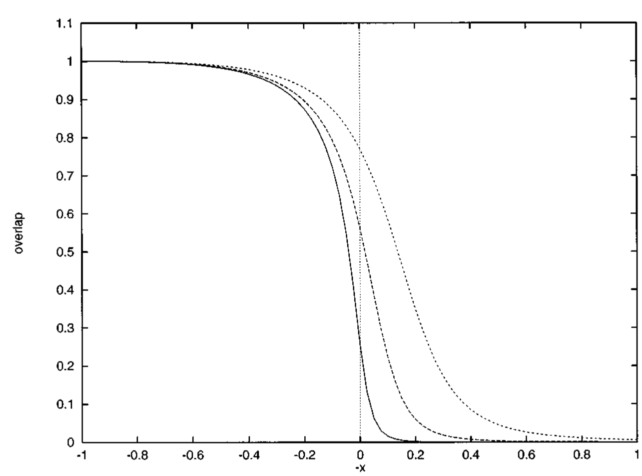

FIG. 2. Same as Fig. 1, with $S=T=0$, except for different values of $g_{\rm ph}$. The solid line corresponds to $g_{\rm ph}=0$, the long dashes to $g_{\rm ph}=1.0g$, and the short dashes to $g_{\rm ph}=2.0g$.

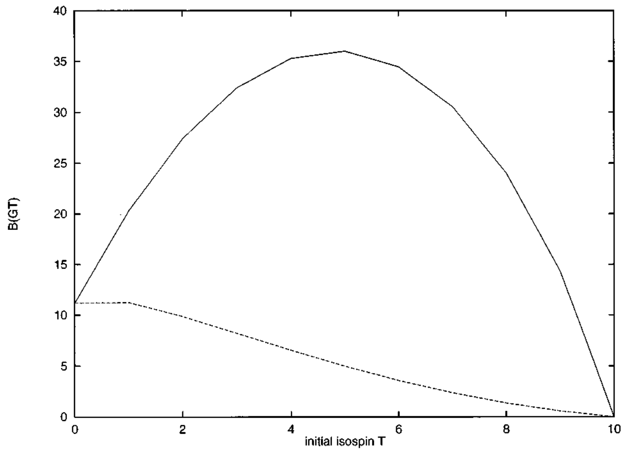

FIG. 3. The Gamow-Teller β^+ strength B(GT) vs the initial neutron excess $T_z = T$ for $\Omega = 12, \mathcal{N} = 10$. The solid curve is for the pure isoscalar spin-triplet pairing phase and the dashed curve for the standard isosvector spin-singlet phase.

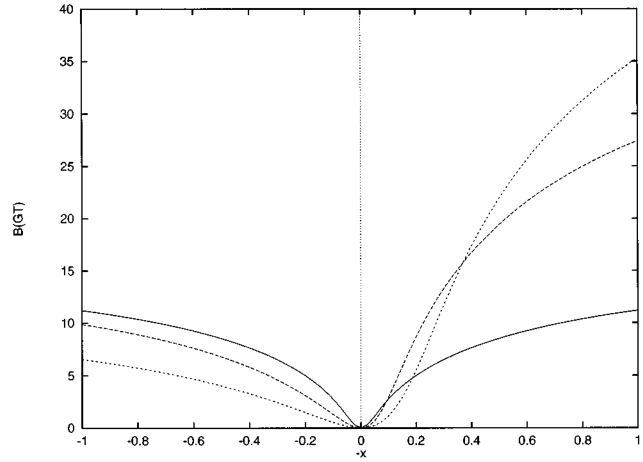

FIG. 4. The Gamow-Teller β^+ strength B(GT) vs the Hamiltonian parameter $-x$, for $\Omega = 12$, $\mathcal{N} = 10$, $S = 0$, and $g_{\mathrm{ph}} = 0$. The solid curve corresponds to $T = 0$, the long dashes to $T = 2$, and the short dashes to $T = 4$.

sition to isoscalar pairing more difficult for the remaining nucleons.

We turn our attention now to transitions induced by the Gamow-Teller (GT) operators $\mathcal{F}^{\mu}_{\nu}$. Their matrix elements are easily evaluated when the wave function is written in the basis $|\mathcal{N}, \lambda, S, T\rangle$ because the operator is diagonal in $\mathcal{N}$ and λ and can change S and T by one unit only. Explicit formulas can be constructed from the SU(4)/SO(4) Clebsch-Gordan coefficients derived in Ref. [14]. The GT operators either increase or decrease $N - Z$ (assumed to be non-negative); the corresponding strengths are called β^+ and β^-. The two strengths are constrained by the Ikeda sum rule

$$S(\beta^-) - S(\beta^+) = 3(N - Z). \tag{4}$$

In the SU(4) limit the β^+ strength vanishes and one state exhausts the β^- strength. In the two extreme limits surrounding SU(4), i.e., $x = 1$, $g_{\mathrm{ph}} = 0$ (the isovector pairing phase) and $x = -1$, $g_{\mathrm{ph}} = 0$ (the isoscalar phase) we can derive analytic expressions for $S(\beta^+)$ from the ground state as a function of $T = T_z = (N - Z)/2$, since in those limits the Hamiltonian contains only the generators of an SO(5) subgroup. The β-decay operators break one pair (of either kind), leading to matrix elements between simple SO(5) representations, the properties of which were studied in Ref. [15]. In the spin-singlet isovector phase we find

$$S(\beta^+) = \frac{(\mathcal{N} - T)(T + 1)(2\Omega - \mathcal{N} - T)}{(2T/3 + 1)(\Omega + 1/2)}. \tag{5}$$

This result applies in a single j shell with degeneracy 2Ω as well as in the model discussed here, which necessarily contains at least 2 degenerate levels in the $j - j$ scheme ($j = l \pm 1/2$). The β^+ strength is plotted as a function of T for $\Omega = 12$, $\mathcal{N} = 10$ in Fig. 3. Except for the initial plateau at low T the behavior of the curve is qualitatively similar to that obtained in BCS theory, where the neutron-proton interaction is ignored. The gradual decrease in strength with T is caused by Pauli blocking.

In the isoscalar phase the result is

$$S(\beta^+) = \frac{(\mathcal{N} - T)(T + 1)(2\Omega - \mathcal{N} - T)}{\Omega - T + 1/2}. \tag{6}$$

Here the behavior of the strength as T increases from 0 is surprisingly different (see Fig. 3). The substantial rise at first seems counterintuitive since the neutron excess is increasing. Blocking is not the only factor at work, however. For lower T the effect is overcome by the collective behavior of the bosonlike S pairs in the final state.

In Fig. 4 we examine the behavior of the strength $S(\beta^+)$ between the two limits, as a function of $-x$ for fixed $g_{\mathrm{ph}} = 0$. [When $g_{\mathrm{ph}} \neq 0$ all curves become flatter, because the system is closer to the SU(4) limit.] As $-x$ increases, the strength $S(\beta^+)$ decreases, vanishing when the SU(4) limit is reached and increasing again as the isoscalar pairing phase is approached until finally it is considerably larger than in the isovector phase. The large strength is caused in part by the transfer of protons from pp pairs, which cannot participate in β^+ decay, to isoscalar np pairs, which can. Only close to $x = -1$, however are this effect and the parabolic isospin dependence fully present; when $-x$ is small the strength can be small as well and the isospin dependence complicated, as is apparent from the crossings of curves in Fig. 4. A large β^+ strength (compared, e.g., to the Ikeda sum rule) therefore reflects very strong isoscalar pairing. If real, it would have important consequences for r-p process nucleosynthesis.

III. DOUBLE-β DECAY, THE QRPA, AND THE RQRPA

We have stressed β^+ strength because of its simplicity and sensitivity to details of nuclear structure. Now, however, we want to discuss modifications of the QRPA, the most frequent and controversial application of which is to double-β decay. Actually, in a model with as few states as this one, the β^+ strength from the ''final'' nucleus (f) essentially determines the double-β decay matrix element (considered here in the closure approximation for simplicity and because the energy denominator can change without a concommitant change in the wave functions). The reason is that the matrix element has the form

$$M_{\mathrm{GT}}^{2\nu}(cl)=\langle 0_f^+|\sum_{i,j}\vec{\sigma}(i)\cdot\vec{\sigma}(j)\tau(i)_-\tau(j)_-|0_i^+\rangle \tag{7}$$

and for a moderate neutron excess the β^- strength, the other relevant quantity, hardly varies with x. In realistic calculations, the QRPA, which has many desirable features, suffers an unfortunate instability when $g_{pp}\equiv(1-x)g/2$ becomes too large that manifests itself through infinite values for both the β^+ strength and the double-β-decay rate. A number of remedies have been proposed recently. One that has received particular attention is the renormalized QRPA (RQRPA) [5,6], which eliminates the instability of the QRPA through a self-consistent calculation of the ground state. The model presented here is ideal for examining how both QRPA and the new approximation work.

The p-n QRPA, described, for example, in Ref. [13] and applied to SO(8) in the same paper, begins with the ordinary BCS ansatz, a coherent state of isovector neutron-neutron and proton-proton pairs, and proceeds by admixing neutron-proton quasiparticle pairs into the ground and excited states. In SO(8) the procedure leads to 2 by 2 matrix equations in each of the two (Fermi/isovector pairing and Gamow-Teller/isoscalar pairing) channels and can be solved by simple diagonalization. More specificially, the one excited state in each channel ($S=0$ or $S=1$) is written in the form

$$|S\rangle=(X_S[\alpha_p^\dagger\alpha_n^\dagger]^{L=0,S}-Y_S[\alpha_p\alpha_n]^{L=0,S})|\tilde{0}\rangle, \tag{8}$$

where $\alpha_p^\dagger$, $\alpha_n^\dagger$ (α_p, α_n) create (destroy) proton and neutron quasiparticles, the brackets indicate angular momentum coupling, $|\tilde{0}\rangle$ is the QRPA ground state, and (X_0,Y_0) and (X_1, Y_1) are the ''physical'' eigenvectors in the spin-0, isospin-1 (Fermi), and spin-1 isospin-0 (Gamow-Teller) channels (additional details are in Ref. [13]). The two channels decouple and for two-neutrino double-β decay only the second is relevant. Associated with the $S=1$ eigenvector is an eigenvalue that becomes complex when g_{pp} reaches a critical value connected with the impending phase transition. The states with complex eigenvalues are not normalizable and have no physical significance, so that the approximation fails to give even an incorrect answer beyond the critical point. This is the ''collapse'' referred to above and is preceded by rapid changes in the β^+ and double-β-decay amplitudes.

In the RQRPA, described for charge-changing modes in Refs. [5,6], the two channels are coupled in an attempt to make the vacuum self-consistent and the resulting equations are nonlinear. For the SO(8) model the equations have seven variables: the two sets of X's and Y's, the eigenvalue associated with each set, and a renormalization parameter. The iterative procedure advocated in Ref. [5] often does not converge here, but the model's simplicity makes the equations easy to solve by other means. Unlike the QRPA, the RQRPA never exhibits the analog of the complex eigenvalues that signal instability, and therefore never yields rapidly changing matrix elements. The question is whether any important physics is lost in the process of guaranteeing a ground state that is built on the BCS state.

Figure 5 presents the exact β^- and β^+ strengths for fixed $\mathcal{N}$ and Ω and several values of T, along with the QRPA and RQRPA approximations to the strengths. The QRPA breakdown is reflected in the expression for the β^+ strength, which blows up at the critical value of x (or g_{pp}). By contrast the RQRPA strength is perfectly stable. The graphs make it clear, however, that the stability is achieved at a significant price; the very real phase transition to a ground state dominated by isoscalar pairing correlations changes the behavior of $S(\beta^+)$, causing the QRPA to break down, but refuses to show itself at all in the RQRPA approximation. The reason is that in preserving (self-consistently) the basic QRPA ansatz the RQRPA limits the isoscalar correlations in the ground state. Put another way, the QRPA breaks down for a reason; there really is a phase transition and it really is nearby, and the RQRPA erases all traces of it. Thus at the very point at which the QRPA fails the RQRPA also begins to deviate badly from the exact result. To make matters worse, and this has been noted elsewhere [16], renormalization destroys one of the nicest features of the QRPA, the preservation of the Ikeda sum rule Eq. (4). In this model, at least, nothing is gained by using the RQRPA.

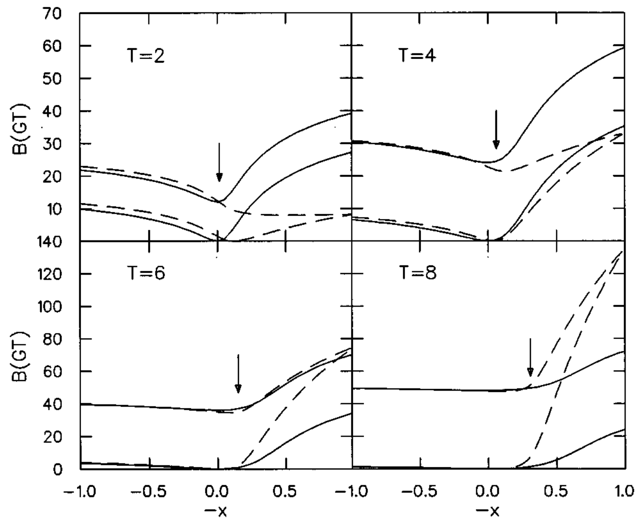

FIG. 5. The Gamow-Teller strength B(GT) for β^- (upper parts of each panel) and β^+ (lower parts of each panel) vs $-x$. The exact results are denoted by the solid lines and the RQRPA results by the dotted lines. Calculated for $\Omega=12$, $\mathcal{N}=10$, $S=0$, and $g_{\mathrm{ph}}=g$ and several values of the isospin T labeling the corresponding panels.

To demonstrate this explicitly for double-β decay, we show in Fig. 6 the matrix element $M_{\mathrm{GT}}^{2\nu}$ for $\mathcal{N}=12$ and $T=4$ as a function of g_{pp}/g_{pair}, where $g_{\mathrm{pair}}=(1+x)/2$. We use this parameter rather than x because it more closely resembles that used in realistic calculations. We have set $g_{\mathrm{ph}}=1.5g$ so that the QRPA breaks down just beyond the point at which the matrix element crosses the origin [when $g_{pp}=g_{\mathrm{pair}}$ i.e., at the SU(4) point]. This is the situation in more realistic calculations as well, but our model shows it to be pure coincidence; the breakdown of the QRPA moves to larger g_{pp} as the essentially independent parameter g_{ph} is increased, while the crossing point never moves, implying that nothing fundamental is behind the proximity of the crossing to the point at which the QRPA fails in realistic calculations. Interestingly, the exact matrix element in our model varies smoothly as the phase transition is traversed, in

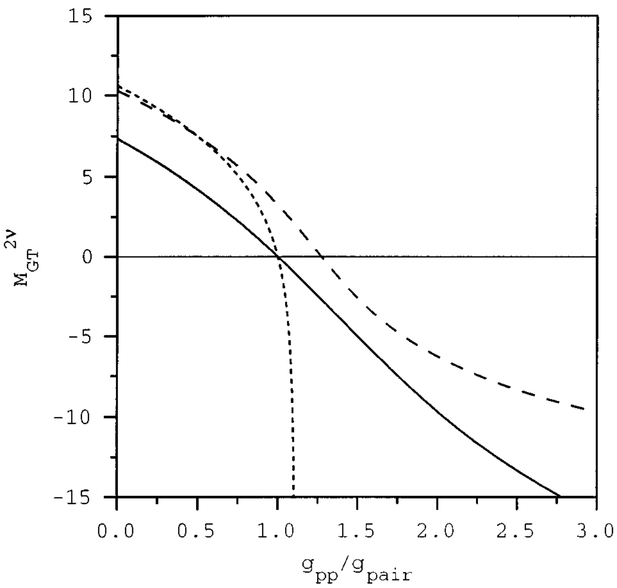

FIG. 6. Double-β decay matrix element $M_{\mathrm{GT}}^{2\nu}$ vs g_{pp}/g_{pair}. The exact results are denoted by the solid line, the RQRPA results by the dashed line, and the QRPA results by the dotted line. Calculated for $\Omega=12$, $\mathcal{N}=12$, $S=0$, $T=4$, and $g_{\mathrm{ph}}=1.5g$.

sharp contrast to the rapid drop in the prediction of the QRPA, which as noted blows up completely just past the crossing point. The RQRPA, as in the last figure, begins to fail at the same point and offers little advantage over the QRPA itself.

In more realistic calculations this last conclusion may or may not hold. The simple model examined here contains only two very collective degrees of freedom. It is certainly possible that with less collectivity the QRPA approximation is worse and the breakdown occurs far from the actual phase transition. In that event the RQRPA would offer advantages, especially in the region between the breakdown of the QRPA and the real phase transition. It would therefore be useful to examine the approximation in a model that dilutes the collectivity of the $T=0$ np pairs but is still solvable. A two-level version of SO(8), i.e., $\mathrm{SO}(8)\times\mathrm{SO}(8)$ [10], might be a good place to start. Here we can say only that we find no evidence supporting the validity of the RQRPA.

IV. APPLICATION OF ''GENERALIZED BCS THEORY''

In this section, we apply generalized pairing theory to the SO(8) model, to assess its ability to provide a meaningful approximate description of the ground-state dynamics of the model in the various phases. We simplify the model slightly by setting g_{ph} to zero (and g to 2, which merely scales the energies).

Generalized pairing theory is well reviewed in Ref. [2] and thus will not be discussed in detail here. Suffice it to say that the theory is founded in the Hartree-Fock-Bogoliubov (HFB) approximation, supplemented by the further assumption that the only nonzero matrix elements of the Hartree Fock and pair potentials are those connecting the four states $|\alpha p\rangle$, $|\alpha n\rangle$, $|\bar{\alpha} p\rangle$, and $|\bar{\alpha} n\rangle$, where $|\bar{\alpha}\rangle$ denotes the state obtained by time-reversal on the state $|\alpha\rangle$. As such, the theory naturally accommodates all pairing modes on an equal footing. This includes the usual $p\bar{p}$ and $n\bar{n}$ pairing as well as $p\bar{n}$, $n\bar{p}$, and pn pairing. Note that here and in subsequent discussion we explicitly distinguish the pairing of particles in ''the same orbit'' (e.g., np) from the pairing of particles in ''time-reversed orbits'' (e.g., $n\bar{p}$).

We have chosen to formulate the theory in terms of the density matrix $\rho(\alpha)$ and the pairing tensor $t(\alpha)$. These matrices, after invoking time-reversal invariance, take the parametrized forms

$$\rho(\alpha)=\begin{pmatrix} \rho_1 & \rho_0 e^{-i\theta} & 0 & \rho_3 e^{-i\theta} \\ \rho_0 e^{i\theta} & \rho_2 & -\rho_3 e^{-i\theta} & 0 \\ 0 & -\rho_3 e^{i\theta} & \rho_1 & \rho_0 e^{i\theta} \\ \rho_3 e^{i\theta} & 0 & \rho_0 e^{-i\theta} & \rho_2 \end{pmatrix}_\alpha ,$$

$$t(\alpha)=\begin{pmatrix} 0 & t_3 e^{-i\theta} & t_1 & t_0 e^{-i\theta} \\ -t_3 e^{-i\theta} & 0 & t_0 e^{i\theta} & t_2 \\ -t_1 & -t_0 e^{i\theta} & 0 & t_3 e^{i\theta} \\ -t_0 e^{-i\theta} & -t_2 & -t_3 e^{i\theta} & 0 \end{pmatrix}_\alpha , \quad (9)$$

with the coefficients interrelated by four unitarity conditions

$$(1-\rho_1-\rho_2)\rho_0-(t_1+t_2)t_0=0,$$

$$(1-\rho_1-\rho_2)\rho_3+(t_1+t_2)t_3=0,$$

$$\rho_1-\rho_1^2-\rho_0^2-\rho_3^2-t_0^2-t_1^2-t_3^2=0,$$

$$\rho_2-\rho_2^2-\rho_0^2-\rho_3^2-t_0^2-t_2^2-t_3^2=0. \quad (10)$$

In our application to the SO(8) model, we impose constraints on the average number of neutrons and the average number of protons of the system, thereby fixing the parameters ρ_1 and ρ_2 according to

$$\rho_1=\frac{Z}{2\Omega} \qquad \text{and} \qquad \rho_2=\frac{N}{2\Omega}. \quad (11)$$

Two more parameters, ρ_0 and ρ_3, are fixed from the first two of the unitarity conditions (10).

Our procedure is first to express the energy of the generalized quasiparticle vacuum as a function of the remaining five parameters of the density matrix and pairing tensor and then to look for local minima, rather than to solve the usual self-consistent eigenvalue equation. The remaining two unitarity conditions (10) are implemented via Lagrange multipliers. The system of equations arising from these variational conditions in principle admits several solutions, the energetically lowest of which defines the generalized BCS approximation to the ground state of the system. In this simple model, all solutions correspond to $\theta=0$ or $\pi/2$.

Figure 7 shows the energies associated with the solutions to the generalized BCS equations for the case of $\Omega=12$ and $\mathcal{N}=5$. The results are plotted as a function of the Hamiltonian parameter x and for various values of the neutron number N. The solutions displayed in the figure have the following character.

Solution A: corresponds to pure $p\bar{p}$ and $n\bar{n}$ pairing.

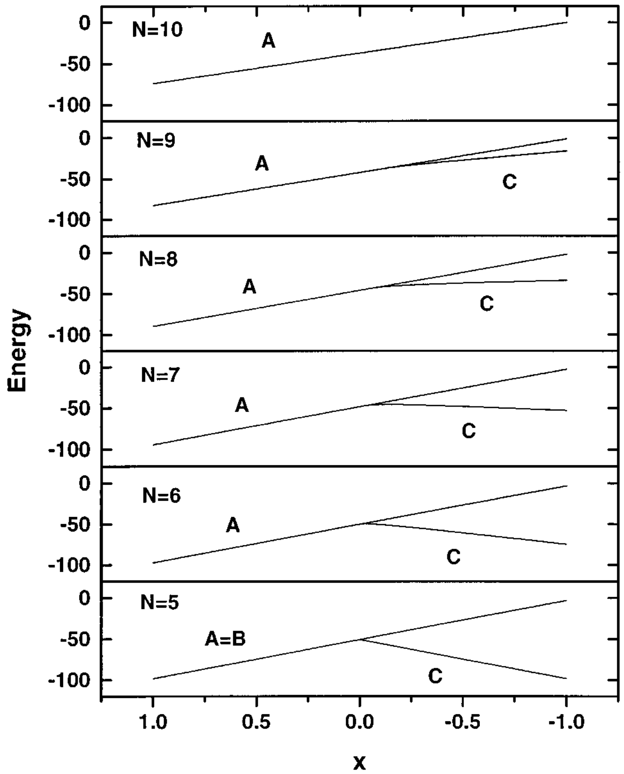

FIG. 7. Energies (in arbitrary units) associated with the different variational solutions to the generalized BCS equations, for $\Omega=12, \mathcal{N}=5$. The panels are labeled by the neutron number N. Here and in Fig. 8 the abscissa is the Hamiltonian parameter x running from 1 to -1. The standard isovector paired state is on the left as in the other figures.

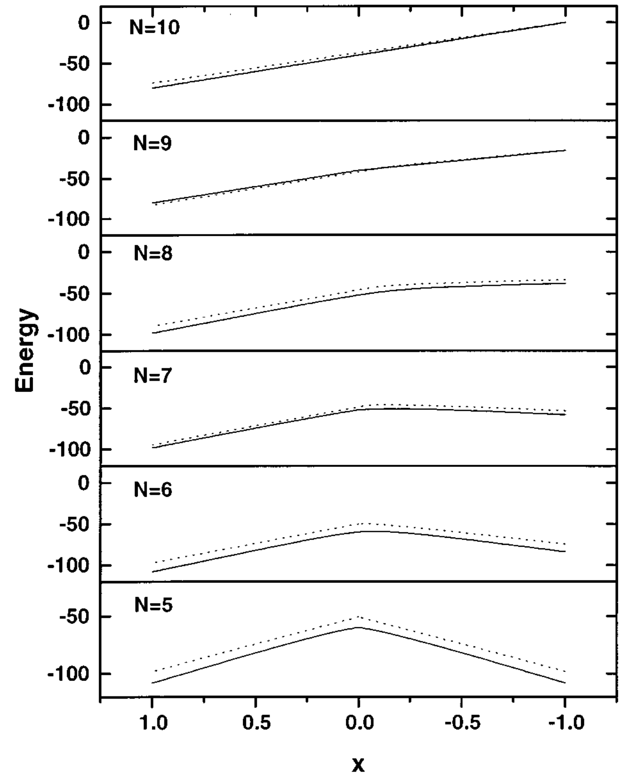

FIG. 8. Comparison of exact ground-state energy (solid curve) with the approximate result obtained in the generalized BCS approximation (dashed curve). The results are shown for $\Omega=12$ and $\mathcal{N}=5$, and for different values of the neutron number N (energies in arbitrary units).

Solution B: corresponds to pure $T=1$ $\overline{pn}$ and $\overline{np}$ pairing. It only exists when $N=Z$, where it is precisely degenerate (but does not mix) with solution A.

Solution C: in general involves both $T=0$ $\overline{pn}$ and $\overline{np}$ pairing and $T=1$ $\overline{pp}$ and $\overline{nn}$ pairing. The relative importance of these different pair correlations is dictated by the three parameters t_0, t_1, and t_2. To a good approximation, the first reflects the number of collective $T=0$ $\overline{pn}$ and $\overline{np}$ pairs, whereas the latter two reflect the number of collective $\overline{pp}$ and $\overline{nn}$ pairs, respectively. As x increases from -1, the solution eventually merges into solution A, ceasing to exist beyond that ''critical point.'' At precisely this point, there is a change in the character of the ground state predicted in generalized BCS approximation that mirrors the true state of affairs.

Figure 8 shows the generalized pairing results for the ground-state energy in comparison with the exact energies discussed earlier, again for $\Omega=12$ and $\mathcal{N}=5$. Included are results corresponding to values of the neutron particle number N ranging from $N=5$ to $N=10$. The results for $N=0$ through $N=4$ follow from the symmetry of the problem. The generalized pairing results correctly reproduce the trends of the exact results, equally well in both phases. The good predictions hold up for even-even nuclei, odd-odd nuclei, symmetric nuclei with $N=Z$, and nuclei with $N=0$ or $Z=0$. The generalized pairing approximation even reproduces the gradual shift of the phase transition to negative values of x when the difference between N and Z increases.

We now turn one last time to Gamow-Teller matrix elements, focusing here (as above) on the summed β^+ strength. We obtain the strength by evaluating the quasiparticle vacuum expectation value of the $1+2$ body operator

$$\hat{S}_{\beta^+}=3\sum_{l_1,m_{l_1},m_{s_1},l_2,m_{l_2},m_{s_2},\mu,\mu}(-)^{\mu}(1\mu\,\tfrac{1}{2}\,m_{s_1}|\,\tfrac{1}{2}\,m_{s_1}+\mu)$$
$$\times(1-\mu\,\tfrac{1}{2}\,m_{s_2}|\,\tfrac{1}{2}\,m_{s_2}-\mu)p^{\dagger}_{l_2,m_{l_2},m_{s_2}-\mu}\,n_{l_2,m_{l_2},m_{s_2}}$$
$$\times n^{\dagger}_{l_1,m_{l_1},m_{s_1}+\mu}\,p_{l_1,m_{l_1},m_{s_1}}. \tag{12}$$

Here $p^{\dagger}(p)$ creates (annihilates) a real proton and $n^{\dagger}(n)$ creates (annihilates) a real neutron. When the ground state is dominated by ordinary $\overline{pp}$ and $\overline{nn}$ pairing and represented by solution A, the total β^+ strength is given by the usual formula

$$S_{\beta^+}=3Z-\frac{3NZ}{2\Omega}. \tag{13}$$

When solution C applies, the result is

$$S_{\beta^+}=3Z-\frac{3NZ}{2\Omega}+4\Omega^2\rho_0^2. \tag{14}$$

(The β^- strengths follow from the above and the Ikeda sum rule, which is preserved at the BCS level.) In Fig. 9, we compare the exact [Eqs. (5) and (6)] and generalized BCS

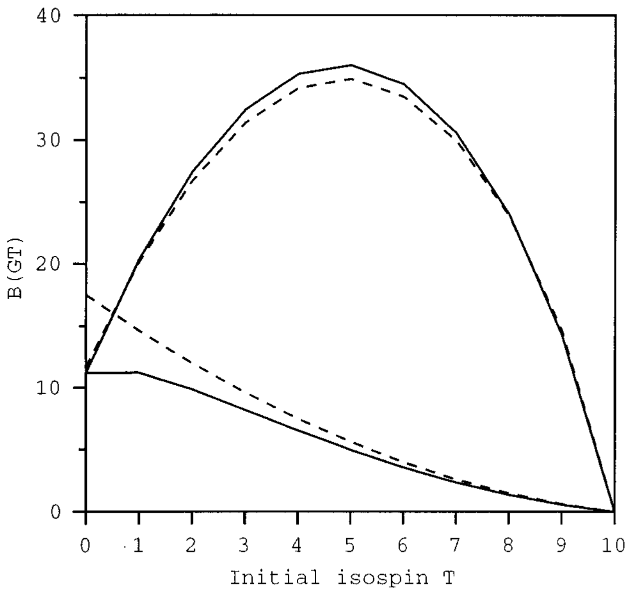

FIG. 9. The Gamow-Teller β^+ strength B(GT) for $\Omega=12$ and $\mathcal{N}=10$ as a function of the isospin T in the two limiting phases. Solid lines represent the exact solutions and dashed lines the generalized BCS solutions in the isovector (bottom two lines) and isoscalar (top two lines) phases.

results for the summed β^+ strength as a function of T_z in the isovector and isoscalar limits when $\Omega=12$ and $\mathcal{N}=10$. Solution A, which is equivalent to the ordinary BCS product wave function, reproduces the general trends of the exact β^+ strength but differs significantly as $T_z\rightarrow 0$. The results in the isoscalar limit are much better for all values of T_z, including $T_z=0$. Only for energies are the BCS results equally good in both phases.

Figure 10 compares the exact and generalized BCS β^+ strengths for the same Ω and $\mathcal{N}$ values as in Fig. 9. Here, however, we only consider the results for $N=14$ and $Z=6$ $(T_z=4)$, but for all values of x. Once again, the generalized BCS approximation provides a somewhat better reproduction of the trends in the exact results when the system is dominated by isoscalar pairing correlations. The suppression of

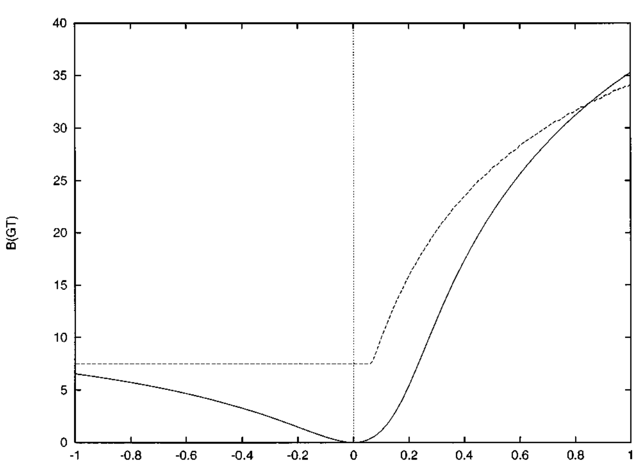

FIG. 10. Exact (solid) and generalized BCS (dashed) Gamow-Teller β^+ strength B(GT) vs the Hamiltonian parameter $-x$ for $\Omega=12$, $\mathcal{N}=10$, and $T_z=4$.

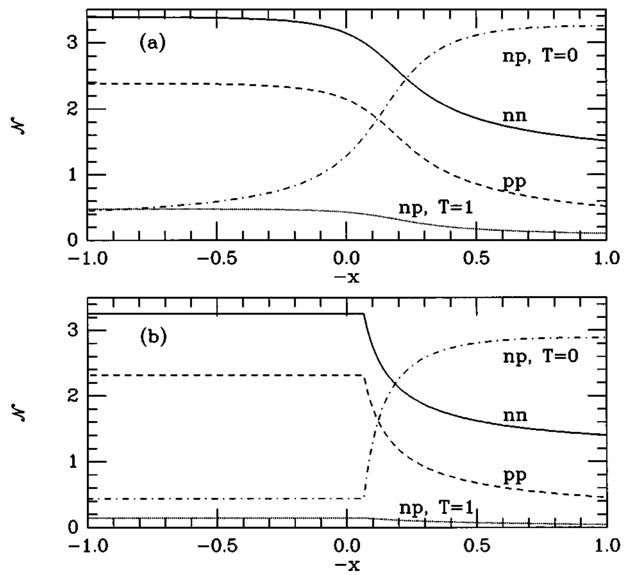

FIG. 11. ''Numbers'' of different types of pairs in the exact (a) and BCS (b) solutions as a function of $-x$, for $\Omega=12$, $\mathcal{N}=10$, and $T_z=4$. The solid line measures the number of nn pairs, the dashed line the number of pp pairs, the dot-dashed line the number of $T=0$ np pairs, and the dotted line the number (nonzero only because of the rough definition of ''number'') of $T=1$ np pairs.

β^+ strength that shows up as the SU(4) limit is approached, however, is not described well by the generalized BCS formalism, either on the isoscalar or isovector side. More accurate results in this regime may require an extension beyond our BCS approximation that fully accommodates the coexistence of all the different pairing modes on both sides of the phase transition.

A striking feature of Fig. 10 is the rise of β^+ strength in the isoscalar region as $-x$ increases to 1. That this rise is correlated as noted earlier with the transfer of nucleons from pp and nn pairs to $T=0$ np pairs (in time reversed orbits — we have dropped the overscore) is apparent from Fig. 11 where we plot the number of collective pairs of different types for the same system as in Fig. 10. We show both the generalized BCS results and the ''exact'' results, using the standard prescription for operators that roughly measure the number of collective pairs [4]. (It is this prescription, which we believe can be improved by a better treatment of Pauli effects, that is responsible for the appearance of low levels of np pairs before the phase transition, where the corresponding pairing tensor is zero.) Part (b) of the figure clearly shows that in BCS theory the wave function does not change from its product form until the phase transition is reached. It also shows that the constant wave function is actually not so bad an approximation when T_z is as large as in the figure. Only when $N\approx Z$ does the theory, which will not reflect np pairing to the left of the phase transition, fail badly. Even then, however, the BCS approximation manages to reproduce energies well.

V. SUMMARY AND CONCLUDING REMARKS

We have examined the interplay between isovector and isoscalar pairing modes in an exactly solvable SO(8) model, focusing on the matrix elements of charge-changing operators. The behavior of the β^+ strength in the isoscalar phase is counterintuitive, rising with increasing neutron excess instead of falling. Double-β decay, by contrast, varies smoothly and predictably on both sides of the phase transition. This behavior is in sharp contrast to the predictions of the QRPA, which fails when isoscalar pairing becomes too strong.

Partly for this reason, we have tested two approximation schemes that purport to better accommodate neutron-proton correlations. One, the RQRPA, works well in the isovector phase but fails completely to capture the physics of the phase transition. Ironically the second approximation, provided by generalized BCS theory, does a better job for the total β-decay strength in the isoscalar phase than in the isovector phase. (It successfully reproduces ground-state energies everywhere, however, even in the vicinity of the transition.) The reason is that the strength operator is a scalar in space and spin, and is therefore not sensitive to the spin ''deformation'' that inhabits the BCS wave function in the isoscalar phase. On the other hand, the isospin deformation in the isovector phase distorts the expectation value of the strength operator, which contains isoscalar, isovector, and isotensor pieces. In Ref. [4] it was shown in a simpler model, based on SO(5) and containing only isovector pairing interactions, that projection of the generalized BCS quasiparticle vacuum onto states with good isospin after variation can fix this problem (though there the analog of solution B was used as the ''isointrinsic state''). It is far from clear, however, that projection will allow the dynamical mixing of isoscalar and isovector pairing before the phase transition is reached. Recently [17], the Lipkin-Nogami method was shown to do the trick at least in part, and it would be interesting to test it in a model like this one. On the other hand, the phases are mixed even at the BCS level on the isoscalar side of the critical point.

The shortcomings of the RQRPA and the successes of generalized pairing theory raise the following question: Can the generalized quasiparticle vacuum be used as a starting point for a ''generalized QRPA'' that works even in the region of the phase transition? Something along these lines has been attempted in Ref. [18], but only after forcing the BCS to mix isoscalar and isovector pairing in an artifical way. Our BCS also appears not to mix the two kinds of pairs except to the right of the critical point, a region that is probably unphysical in nuclei that undergo double-β decay. Perhaps a more fruitful approach, therefore, will be a more self-consistent QRPA, in which the RPA and BCS equations are coupled. Such a procedure can rescue the Ikeda sum rule [19] and could conceivably facilitate isoscalar-isovector mixing at the BCS level even to the left of the critical point. Other modifications of the basic BCS procedure, including approximations to projection, may also mix the pairing modes without complicating the method too severely. Which of these approaches is the simplest and most useful remains to be seen.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant Nos. PHY-9303041, PHY-9600445, and INT-9224875, by the U.S. Department of Energy under Grant Nos. DE-FG05-94ER40827 and DE-FG03-88ER-40397, by NATO under Grant No. CRG.900466, by the Bulgarian National Foundation for Scientific Research under Contract No. Φ-527, by the DIGICYT (Spain) under Contract No. PB95/0123, and by the European Union under Contract No. CI1*-CT94-0072. One of the authors (M.V.S.) would like to acknowledge the support of the Fulbright Foundation, and two others (J.E. and P.V.) the hospitality of the Institute for Nuclear Theory at the University of Washington, where some of this work was carried out.

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